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# Uniform asymptotic expansions of integrals: a selection of problems

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# Uniform Asymptotic Expansions of Integrals: A Selection of Problems

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**Abstract**

On the occasion of the conference we mention examples of Stieltjes' work on asymptotics of special functions. The remaining part of the paper gives a selection of asymptotic methods for integrals, in particular on uniform approximations. We discuss several "standard" problems and examples, in which known special functions (error functions, Airy functions, Bessel functions, etc.) are needed to construct uniform approximations. Finally, we discuss the recent interest and new insights in the Stokes phenomenon. An extensive bibliography on uniform asymptotic methods for integrals is given, together with references to recent papers on the Stokes phenomenon for integrals and related topics.

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## 1. Examples of Stieltjes' work in asymptotics

Stieltjes has several results on asymptotic expansions, including discussions of the remainder terms in the expansions. In this section we give examples of his interest in asymptotics of special functions.

### *1.1. Stirling's series for* $\ln\Gamma(z)$

Stieltjes[1]) has considered the well-known result for the logarithm of the Euler gamma function:

$$\ln\Gamma(z) \sim z\ln z - z + \frac{1}{2}\ln(2\pi/z) + + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} + \dots$$

as $z\to\infty, |\arg z| < \pi$. More precisely

$$\ln\Gamma(z) = z\ln z - z + \frac{1}{2}\ln(2\pi/z) + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n\,(2n-1)}\frac{1}{z^{2n-1}} + R_N(z)$$

[1]) T.J. Stieltjes (1889), Sur la développement de $\log\Gamma(a)$, *J. Math.*, Paris, sér. 4, 5, 425–444. Collected papers (1993), Vol. 2, Springer Verlag, Berlin, 215–234.

where $B_n$ are the Bernoulli numbers and

$$R_N(z) = \mathcal{O}\left(\frac{1}{z^{2N-1}}\right), \quad z \to \infty, |\arg z| < \pi.$$

Stieltjes showed that:

$$|R_N(z)| \leq \frac{|B_{2N}|}{2N(2N-1)} \frac{1}{\cos^{2N}(\frac{1}{2}\theta)} \frac{1}{|z|^{2N-1}}$$

where $\theta = \arg z \in (-\pi, \pi)$. A proof of this result can also be found in OLVER (1974), page 294.

*1.2. Bessel functions*

Stieltjes' THÈSE DE DOCTORAT[2]) gives several results on Bessel functions, especially the asymptotic expansions of $J_0(x), Y_0(x), K_0(x)$. One has for the ordinary Bessel functions, with $\chi = z - (\frac{1}{2}\nu + \frac{1}{4})\pi$,

$$\begin{aligned} J_\nu(z) &= \sqrt{\tfrac{2}{\pi z}}\,[P(\nu,z)\cos\chi - Q(\nu,z)\sin\chi], \\ Y_\nu(z) &= \sqrt{\tfrac{2}{\pi z}}\,[P(\nu,z)\sin\chi + Q(\nu,z)\cos\chi], \end{aligned}$$

where $P$ and $Q$ are the non-oscillating parts, having the expansions

$$P(\nu,z) \sim \sum_{n=0}^{\infty}(-1)^n \frac{(\nu,2n)}{(2z)^{2n}}, \quad Q(\nu,z) \sim \sum_{n=0}^{\infty}(-1)^n \frac{(\nu,2n+1)}{(2z)^{2n+1}},$$

as $z \to \infty$ in $|\arg z| < \pi$. The symbols $(\nu, n)$ are given by:

$$\begin{aligned} (\nu,n) &= \frac{2^{-2n}}{n!}(4\nu^2-1)(4\nu^2-3^2)\cdots[4\nu^2-(2n-1)^2] \\ &= \frac{\Gamma(\frac{1}{2}+\nu+n)}{n!\,\Gamma(\frac{1}{2}+\nu-n)}, \qquad n = 0,1,2,\ldots . \end{aligned} \tag{1.1}$$

Let the remainder terms in the expansions of $P(0,x)$ and $Q(0,x)$ be written in the form

$$P(0,x) = 1 - \frac{9}{2!(8x)^2} + \ldots + (-1)^{n-1}\frac{9\cdot 25\cdots(4n-5)^2}{(2n-2)!(8x)^{2n-2}} + \theta_1(-1)^n\frac{9\cdot 25\cdots(4n-1)^2}{(2n)!(8x)^{2n}}$$

$$Q(0,x) = -\frac{1}{1!8x} + \frac{9\cdot 25}{3!(8x)^3} - \ldots + (-1)^n\frac{9\cdot 25\cdots(4n-3)^2}{(2n-1)!(8x)^{2n-1}} + \theta_2(-1)^{n-1}\frac{9\cdot 25\cdots(4n+1)^2}{(2n+1)!(8x)^{2n+1}}.$$

By using a quite ingenious method Stieltjes showed that

$$0 < \theta_1 < 1, \quad 0 < \theta_2 < 1.$$

He also showed that

$$\theta_1, \theta_2 \sim \frac{1}{2} \quad \text{as} \quad x \to \infty,$$

[2]) T.J. Stieltjes (1886), Recherches sur quelques séries semi-convergents (Thèse de doctorat), *Ann. Sci. Éc. norm.*, Paris, sér. 3, 3, 201–258. Collected papers (1993), Vol. 2, Springer Verlag, Berlin, 6–62.

when the expansion is terminated at the smallest term (that is, when $n \sim 2x$).

For the function $K_0(x)$ Stieltjes used the integral

$$K_0(x) = \frac{e^{-x}}{\sqrt{2}} \int_0^\infty e^{-xu} \frac{du}{\sqrt{u}\,\sqrt{1+u/2}},$$

and he showed

$$K_0(x) = \sqrt{\tfrac{\pi}{2x}}\, e^{-x} \left[\sum_{n=0}^{N-1} \frac{(0,n)}{(2x)^n} + \theta \frac{(0,N)}{(2x)^N}\right], \tag{1.2}$$

with $0 < \theta < 1$. His proof was based on the identity:

$$\frac{1}{\sqrt{1+u/2}} = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\frac{1}{2}u\sin^2\phi}.$$

Expansion in powers of $u$ gives

$$\frac{1}{\sqrt{1+u/2}} = \sum_{n=0}^{N-1} \binom{-\frac{1}{2}}{n} \left(\tfrac{1}{2}u\right)^n + (-1)^N \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{(\frac{1}{2}u\sin^2\phi)^N}{1+\frac{1}{2}u\sin^2\phi} d\phi.$$

Obviously

$$(-1)^N \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{(\frac{1}{2}u\sin^2\phi)^N}{1+\frac{1}{2}u\sin^2\phi} d\phi = \theta(-1)^N \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \left(\tfrac{1}{2}u\sin^2\phi\right)^N d\phi = \theta \binom{-\frac{1}{2}}{N} \left(\tfrac{1}{2}u\right)^N,$$

where $\theta$ lies between 0 and 1. Multiplying by $e^{-xu}/\sqrt{u}$ and integrating we obtain the expansion in (1.2) with remainder, and with a bound on the remainder.

Stieltjes methods for the Bessel functions are also treated in [103], §§7.31, 7.32.

*1.3. The logarithmic integral*

In his THÈSE DE DOCTORAT, Stieltjes considered

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t},$$

which for $x > 1$ is a Cauchy principal value integral. He found:

$$\mathrm{li}\left(e^x\right) = \frac{e^x}{x}\left[1 + \frac{1}{x} + \frac{2!}{x^2} + \ldots + \frac{(n-1)!}{x^{n-1}} + R_n\right],$$

with

$$R_n = e^{-x}\sqrt{\frac{2\pi}{n}}\left[A_0 + \frac{A_1}{n} + \frac{A_3}{n^3} + \ldots\right],$$

$x = n + \eta$, and

$$A_0 = \eta - \frac{1}{3}, \quad A_1 = \frac{1}{6}\eta^3 - \frac{1}{2}\eta^2 + \frac{1}{12}\eta + \frac{1}{540}$$

We see that the remainder is exponentially small when we truncate the expansion at the right place (when $n \sim x$). Recently, re-expanding remainders at the right place has received renewed interest; we return to this topic in Section 8.

## 2. Asymptotic expansions of integrals: basic steps

We are concerned with obtaining uniform asymptotics expansions of integrals of the type

$$F_\alpha(z) = \int_{\mathcal{C}} f(t) e^{-z\phi(t,\alpha)}\, dt$$

where $\mathcal{C}$ is a contour in the complex plane and $z$ is a large parameter. For certain values of the other parameter $\alpha$ (the so-called uniformity parameter) the asymptotic behaviour of $F_\alpha(z)$ may change. For instance, in turning point problems, for $\alpha < 0$ the integral $F_\alpha(z)$ may show strong oscillations, whereas for $\alpha > 0$ the integral may be monotonic. For obtaining uniform expansions the following major steps can be distinguished:

- Trace the critical points on or near $\mathcal{C}$ that significantly contribute to $F_\alpha(z)$; that is, points where $\partial\phi/\partial t$ vanishes (saddle points), poles, singularities, or end points of $\mathcal{C}$.
- Investigate the nature and significance of these critical points: which ones give the dominant contributions, and can be reached by deforming $\mathcal{C}$ without disturbing convergence, etc.?
- Transform the integral into a standard form by a conformal mapping, taking into account the mapping of the contour and the critical points.
- Construct a formal expansion by using local expansions at the relevant critical points, Laplace's method, integration by parts, etc.
- Discuss the nature and the asymptotic properties of the expansion.
- Construct error bounds for the remainders.
- Extend the results to wider domains of the parameters, e.g. by deforming $\mathcal{C}$.

The first four points are most frequently the only possibilities in practical problems (wave theory, optics, diffraction and scattering theory); often the contributions in the expansion have a physical interpretation and then just the form of the expansion is the ultimate requirement.

Standard books on asymptotic analysis are [16], [50], and [106], the second one concentrating on differential equations, the other ones on integrals.

## 3. Why uniform asymptotics?

We give well-known examples of non-uniform expansions and indicate that a secondary parameter may disturb the nature of the expansion. The special functions in the examples all have uniform expansions in which the parameters may vary in much larger domains. Uniform expansions are useful in describing the transition of behaviour. In physical or statistical problems this may give insight in the underlying problems. In general, higher transcendental functions are needed to describe the transitions (Airy functions, error functions, parabolic cylinder functions, etc.). Uniform expansions are also useful for numerical evaluations, because more robust algorithms can be developed when uniform expansions are used. Usually, however, coefficients become more complicated in uniform expansions.

*3.1. Incomplete gamma function*
The incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t}\, dt$$

has the well-known asymptotic expansion

$$\Gamma(a,z) \sim z^a e^{-z}\left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \ldots\right]$$

as $z \to \infty$. This expansion can be obtained by integrating by parts. What about $a$? Should $a$ be fixed, or is it possible to take $a = \mathcal{O}(\sqrt{z}\,)$, or larger? In Section 7.4 we present a uniform expansion containing the error function that is valid in the transition area $a \sim z$ (and in a larger domain).

*3.2. Modified Bessel function*
The modified Bessel function

$$K_\alpha(z) = \int_0^\infty e^{-z\cosh t} \cosh \alpha t\, dt$$

has the following asymptotic expansion:

$$K_\alpha(z) \sim \sqrt{\tfrac{\pi}{2z}}\, e^{-z} \sum_{n=0}^\infty \frac{(\alpha, n)}{(2z)^n}, \quad z \to \infty, \quad |\arg z| < \tfrac{3}{2}\pi.$$

The symbols $(\alpha, n)$ are defined in (1.1). What about $\alpha$? Should $\alpha$ be fixed, can we take $\alpha = \mathcal{O}(\sqrt{z}\,)$, or larger? A uniform expansion in terms of elementary functions is given in [50], page 378. The emphasis is on large values of $\alpha$, but the expansion also has an asymptotic property for large $z$. The expansion can also be found in [1], page 378.

*3.3. Probability distribution function*
Many statistical distribution functions can be transformed into the standard form

$$F_a(x) = \int_{-\infty}^x e^{-at^2} f(t)\, dt.$$

The asymptotic behaviour of $F_a(x)$ as $a \to \infty$ depends strongly on sign$(x)$. Assume $F_a(\infty) = 1$. Then as a rule

$$F_a(x) = \begin{cases} -f(x)/(2ax)e^{-ax^2}[1 + \mathcal{O}(1/a)], & \text{if } x < 0; \\ \tfrac{1}{2} f(0)[1 + \mathcal{O}(1/\sqrt{a}\,)], & \text{if } x = 0; \\ 1 + \mathcal{O}(1/a), & \text{if } x > 0. \end{cases}$$

Conclusion: the asymptotic behaviour of $F_a(x)$ is completely different in the three cases distinguished and, moreover, the asymptotic forms do not pass into one another when $x$ changes from negative values to positive ones. A uniform approximation can be given in terms of the *error function*; see §5 and [85].

*3.4. Normalized incomplete gamma function*
As a special case of the previous subsection we consider the normalized incomplete gamma function

$$Q(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1} e^{-t}\, dt$$

When $a$ is large and $x > a$ the function $Q(a,x)$ is very small; when $x$ becomes smaller and crosses the value $a$ there is a sudden change in behaviour form 0 to 1.

We can define a transformation $t \mapsto \tau(t)$, such that the *gamma distribution* becomes a *Gaussian distribution*. This gives

$$Q(a,x) = \int_\zeta^\infty e^{-a\tau^2} f(\tau)\, d\tau,$$

which is a standard form for this kind of problem. A uniform approximation can be given in terms of the error function, as will be explained in §7.4.

## 4. Laplace integrals and Watson's Lemma

The first step in the asymptotics of integrals is Watson's lemma. Consider

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-z\,t} f(t)\, dt.$$

Expansion at the origin $f(t) = \sum_{n=0}^\infty c_n t^n$ gives

$$F_\lambda(z) \sim \sum_{n=0}^\infty c_n \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} z^{-n-\lambda}$$

as $z \to \infty$, $\lambda$ fixed; see [50],page 113 and further, also for more general expansions of $f$. When $\lambda$ is not fixed (say, $\lambda$ is depending on $z$) this expansion becomes invalid. Let $\mu := \lambda/z$. When $\mu$ is not small, it is better to expand at $t = \mu$. The expansion $f(t) = \sum_{n=0}^\infty a_n(\mu)\ (t-\mu)^n$ gives

$$F_\lambda(z) \sim \sum_{n=0}^\infty a_n(\mu)\ P_n(\lambda)\ z^{-n-\lambda},$$

where $P_n(\lambda)$ are polynomials,

$$P_0(\lambda) = 1, \quad P_1(\lambda) = 0, \quad P_2(\lambda) = \lambda, \quad P_3(\lambda) = 2\lambda, \ldots$$

with recursion relation

$$P_{n+1}(\lambda) = n\left[P_n(\lambda) + \lambda P_{n-1}(\lambda)\right].$$

It is easily seen that

$$P_n(\lambda) = \mathcal{O}\left(\lambda^{[n/2]}\right), \quad \text{as} \quad \lambda \to \infty.$$

Under mild conditions on $a_n(\mu)$, that is, on $f$, this expansion is uniformly valid with respect to $\lambda \in [0,\infty)$, and in a larger domain of the complex plane. The main condition on $f$ is that its singularities are not too close to the point $t=\mu$. Let $R_\mu$ denote the radius of convergence of the Taylor expansion of $f$ at $t=\mu$. Then we require

$$R_\mu^{-1} = \mathcal{O}[(1+\mu)^{-\kappa}],\ \mu \ge 0,\ \kappa \ge \frac{1}{2},\ \kappa \text{ fixed}.$$

For more details we refer to [86] and [87].

*4.1. More general Laplace integrals*

A further step is to consider

$$F_\lambda(z) := \int_0^\infty q(t)^{\lambda-1} e^{-z\,p(t)}\,dt, \tag{4.1}$$

$p$ and $q$ analytic, and increasing functions for positive values of $t$, $q(0)=0$. The problem is to find an asymptotic expansion of $F_\lambda(z)$ as $z\to\infty$, which is uniformly valid with respect to $\lambda\in[0,\infty)$. As in Watson's lemma, the basic approximant is the Euler gamma function

$$z^{-\lambda}\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-zt}\,dt, \quad \Re\lambda, \Re z > 0.$$

Examples are:

Laplace integral (standard form):

$$\int_0^\infty t^{\lambda-1}e^{-z\,t}f(t)\,dt,$$

Modified Bessel function:

$$\int_0^\infty [t\,(1+t)]^{\lambda-1}e^{-z\,t}\,dt,$$

Whittaker function:

$$\int_0^\infty [t\,(1+t)^\nu]^{\lambda-1}\,e^{-z\,t}\,dt,$$

Parabolic cylinder function:

$$\int_0^\infty t^{\lambda-1}e^{-z[t+\alpha t^2]}\,dt,$$

Beta integral:

$$\frac{\Gamma(z)\Gamma(\lambda)}{\Gamma(z+\lambda)} = \int_0^1 t^{z-1}(1-t)^{\lambda-1}\,dt = \int_0^\infty \left(1-e^{-t}\right)^{\lambda-1} e^{-z\,t}\,dt.$$

In the general case

$$F_\lambda(z) := \frac{1}{\Gamma(\lambda)}\int_0^\infty q(t)^{\lambda-1}e^{-z\,p(t)}\,dt = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-z[p(t)-\mu\ln q(t)]}\,\frac{dt}{q(t)}, \quad \mu=\frac{\lambda}{z},$$

we determine the saddle point $t_0$, that is, the solution of $p'(t)-\mu q'(t)/q(t)=0$. The transformation to the standard form reads

$$p(t)-\mu\ln q(t) = \tau - \mu\ln\tau + A(\mu).$$

The condition on the conformal mapping is that the saddle point at $t = t_0$ maps to the saddle point at $\tau = \mu$, and this determines $A(\mu)$. Under certain conditions on $p, q$ this gives the standard form:

$$F_\lambda(z) = \frac{e^{-zA(\mu)}}{\Gamma(\lambda)} \int_0^\infty \tau^{\lambda-1} e^{-z\tau} f(\tau)\, d\tau,$$

$$f(\tau) = \frac{\tau}{q(t)} \frac{dt}{d\tau} = \frac{\tau - \mu}{q(t)p'(t) - \mu q'(t)}.$$

Again:

$$f(t) = \sum_{n=0}^\infty a_n(\mu)\, (t-\mu)^n \quad \Rightarrow \quad F_\lambda(z) \sim e^{-zA(\mu)} \sum_{n=0}^\infty a_n(\mu)\, P_n(\lambda)\, z^{-n-\lambda},$$

as $z \to \infty$, uniformly valid with respect to $\mu \in [0, \infty)$. An integration by parts procedure gives a slightly better expansion. For details we refer to [87].

## 5. Statistical distribution functions

Many cumulative distribution functions can be transformed into the standard form:

$$F_a(\eta) = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\eta} e^{-\frac{1}{2}a\zeta^2} f(\zeta)\, d\zeta, \tag{5.1}$$

with $f$ analytic in a neighbourhood of $\mathbb{R}$. The problem is to obtain an asymptotic expansion as $a \to \infty$, that holds uniformly with respect to $\eta \in \mathbb{R}$. The saddle point at the origin may be inside or outside the interval of integration. Especially the transition area $\eta \sim 0$ is of interest. In terminology of asymptotics: a saddle point may cross the end-point of integration. This type of distribution function occurs frequently; in fact the gamma distribution, the beta distribution and many other cumulative distribution functions can be transformed into the standard form (5.1).

The basic approximant is the normal distribution function (error function)

$$P(\eta\sqrt{a}\,) = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\eta} e^{-\frac{1}{2}a\zeta^2}\, d\zeta = \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{a/2}\,).$$

The procedure to obtain a uniform expansion is based on integration by parts. We normalize $f(0) = 1$ and write $f(\zeta) = 1 + [f(\zeta) - 1]$. Then

$$\begin{aligned} F_a(\eta) &= P(\eta\sqrt{a}\,) - \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\eta} \frac{f(\zeta)-1}{\zeta}\, d\, e^{-\frac{1}{2}a\zeta^2} \\ &= P(\eta\sqrt{a}\,) + \frac{1}{\sqrt{2\pi a}} \frac{1-f(\eta)}{\eta} e^{-\frac{1}{2}a\eta^2} + \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\eta} e^{-\frac{1}{2}a\zeta^2} f_1(\zeta)\, d\zeta, \end{aligned}$$

where

$$f_1(\eta) = \frac{d}{d\eta} \frac{f(\eta)-1}{\eta}.$$

Repeating this process, we obtain

$$F_a(\eta) = P(\eta\sqrt{a})A(a) \; + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} B_\eta(a),$$

with

$$A(a) \sim \sum_{n=0}^{\infty} \frac{A_n}{a^n}, \quad B_\eta(a) \sim \sum_{n=0}^{\infty} \frac{B_n(\eta)}{a^n},$$

where the coefficients are given by

$$A_n = f_n(0), \quad B_n(\eta) = \frac{f_n(0) - f_n(\eta)}{\eta}, \quad f_n(\eta) = \frac{d}{d\eta} \frac{f_{n-1}(\eta) - f_{n-1}(0)}{\eta}.$$

For more details and applications to several distribution functions we refer to [85].

## 6. Examples of standard forms

In Table 6.1 we give standard forms of integrals for which well-known special functions are used as basic approximants. We give the critical points, the coalescence of which causes uniformity problems. We also give references to the literature.

### Remarks

[1]. We assume that the functions $f$ are analytic in a neighbourhood of the path of integration and at the critical points.
[2]. The integrals reduce to their approximants when $f = 1$.
[3]. Different intervals of integration are also investigated.
[4]. In (6.5) and (6.8) two saddle points coalesce when $\alpha = 0$; the cases are different, because in (6.8) an additional critical point at $t = 0$ is present (pole of the function in the exponent and end point of integration).
[5]. In all cases elementary approximations can be used for fixed values of the parameter $\alpha$.
[6]. Several cases need further investigations with respect to the construction of error bounds and the determination of maximal regions of validity.

### References to the Bibliography

We give references to the literature for the standard forms mentioned in Table 6.1. In many cases the integrals are different from those in the table, but the asymptotic features may be the same in the given references.

[1]. (6.1): [21], [32], [36], [44], [80], [102].
[2]. (6.2): [82].
[3]. (6.3): [32], [55], [62], [77], [83], [85], [90].
[4]. (6.4): [15], [16], [22], [41], [50], [56], [61], [65], [66], [67], [71], [74], [76], [104], [106].
[5]. (6.5): [20], [29], [30], [45], [48], [50], [66], [73], [75], [91], [96], [101], [106].
[6]. (6.6): [2], [69], [86], [87], [94].
[7]. (6.7): [68], [81], [92], [95], [107].
[8]. (6.8): [18], [26], [28], [29], [33], [63], [84], [91], [93].
[9]. (6.9): [27], [70], [99], [101], [105].
[10]. (6.10): [108].

| | Standard Form | Approximant | Critical points |
|---|---|---|---|
| (6.1) | $\int_{-\infty}^{\infty} e^{-zt^2}\frac{f(t)}{t-i\alpha}\,dt$ | Error function | $t=0,\quad t=i\alpha$ |
| (6.2) | $\int_{-\infty}^{\infty} e^{-zt^2}\frac{f(t)}{(t-i\alpha)^{\mu}}\,dt$ | Weber function | $t=0,\quad t=i\alpha$ |
| (6.3) | $\int_{-\infty}^{\alpha} e^{-zt^2} f(t)\,dt$ | Error function | $t=0,\quad t=\alpha$ |
| (6.4) | $\int_0^{\infty} t^{\beta-1}e^{-z(\frac{1}{2}t^2-\alpha t)}f(t)\,dt$ | Weber function | $t=0,\quad t=\alpha$ |
| (6.5) | $\int_{\mathcal{L}} e^{z(\frac{1}{3}t^3-\alpha t)}f(t)\,dt$ | Airy function | $t=\pm\sqrt{\alpha}$ |
| (6.6) | $\int_0^{\infty} t^{\alpha-1}e^{-zt}f(t)\,dt$ | Gamma function | $t=0,\quad t=\alpha/z$ |
| (6.7) | $\int_{\alpha}^{\infty} t^{\beta-1}e^{-zt}f(t)\,dt$ | Incomplete gamma function | $t=0,\quad t=\alpha$ |
| (6.8) | $\int_0^{\infty} t^{\beta-1}e^{-z(t+\alpha/t)}f(t)\,dt$ | Bessel function | $t=0,\quad t=\pm\sqrt{\alpha}$ |
| (6.9) | $\int_{\alpha}^{\infty} e^{-zt}(t^2-\alpha^2)^{\mu}f(t)\,dt$ | Bessel function | $t=\pm\alpha$ |
| (6.10) | $\int_0^{\infty}\frac{\sin z(t-\alpha)}{t-\alpha}f(t)\,dt$ | Sine integral | $t=0,\quad t=\alpha$ |

**Table 6.1.** Standard forms of integrals

[11]. Papers with other special functions as main approximants: [17], [22], [23], [24], [25], [35], [38], [39], [40], [42], [56], [72], [73], [89], [78], [79], [97], [97], [98], [100].
[12]. Monographs on asymptotics are [16], [50], [106].
[14]. Papers with overviews and general aspects are [60], [64], [66], [88].

## 7. Further examples

We consider a few recent examples of uniform expansions for particular special functions. We give a few details of the basic steps for deriving the results. More details can be found in the literature.

*7.1. Laguerre polynomials*

This important example is considered in [29]. The Laguerre polynomials are defined by

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

and have the generating function:

$$(1-t)^{-\alpha-1} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n, \quad |t| < 1.$$

Let

$$\kappa = n + \tfrac{1}{2}(\alpha + 1)$$

Then we can distinguish five $x-$intervals with different asymptotic behaviour for $L_n^\alpha(x)$ when $n$ is large and $\alpha$ is fixed:

$(i)$ $x < 0$: monotonic region,
$(ii)$ $x$ near 0: transition region,
$(iii)$ $0 < x < 4\kappa$: oscillatory region,
$(iv)$ $x$ near $4\kappa$: turning point region,
$(v)$ $x > 4\kappa$: monotonic region.

In cases $(i)$, $(iii)$, $(v)$ we can give expansions in terms of elementary functions. For the transitions from

$(i)$ to $(iii)$ we need a Bessel function,
$(iii)$ to $(v)$ we need an Airy function.

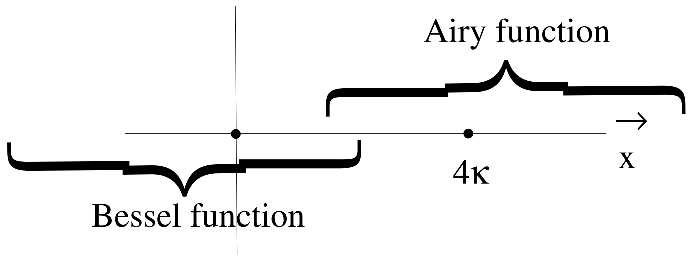


**Figure 7.1.** The $n$ zeros of $L_n^\alpha(x)$ occur in the interval $(0, 4\kappa)$.

When $\alpha$ is fixed, two uniform expansions suffice to cover the real $x-$axis with overlapping domains. In [29] the generating function is used to write the Laguerre polynomial as a Cauchy integral, and then the saddle point method is applied to obtain expansions in terms of Airy and Bessel functions.

**Expansion in terms of Bessel functions.** Let

$$A(x) = \begin{cases} \frac{1}{2}i[\sqrt{x^2-x} - \operatorname{arcsinh}\sqrt{-x}\,], & \text{if } x \le 0; \\ \frac{1}{2}[\sqrt{x-x^2} + \arcsin\sqrt{x}\,], & \text{if } 0 \le x < 1, \end{cases} \tag{7.1}$$

where for $x \in (0,1]$ the quantity $A(x)$ is defined in (7.1). Then

$$2^\alpha e^{-2\kappa x} L_n^\alpha(4\kappa x) \sim \frac{J_\alpha(\nu A)}{A^\alpha} \sum_{k=0}^\infty \frac{\alpha_{2k}}{(2\kappa)^{2k}} - \frac{J_{\alpha+1}(\nu A)}{A^{\alpha+1}} \sum_{k=0}^\infty \frac{\beta_{2k+1}}{(2\kappa)^{2k+1}}$$

as $n \to \infty$, uniformly for $x \in (-\infty, a]$, where $a \in (0,1)$. $J_\alpha(z)$ is the familiar Bessel function. The coefficients $\alpha_{2k}$, $\beta_{2k+1}$ follow from recursions. Let

$$h(u) = \left[\frac{u}{\sinh z(u)}\right]^{\alpha+1} \frac{dz}{du},$$

where the relation between $z$ and $u$ is given by the equation

$$z - x\coth z = u - A^2(x)/u.$$

Define a set of functions $\{h_k\}$, $\{g_k\}$ and coefficients $\{\alpha_k\}$, $\{\beta_k\}$ by writing $h_0(u) = h(u)$ and when $k \ge 1$

$$h_k(u) = \alpha_k + \beta_k/u + (1 + A^2/u^2)g_k(u), \tag{7.2}$$

$$h_{k+1}(u) = g_k'(u) - \frac{\alpha+1}{u} g_k(u).$$

The coefficients $\alpha_k, \beta_k$ are computed by substitution of $u = \pm iA$ in (7.2); $A$ is defined in (7.1).

**Expansion in terms of Airy functions.** Let

$$B(x) = \begin{cases} i[3\beta(x)/2]^{1/3}, & \text{if } 0 < x \le 1; \\ [3\gamma(x)/2]^{1/3}, & \text{if } x \ge 1, \end{cases} \tag{7.3}$$

$$\beta(x) = \frac{1}{4}\pi - A(x) = \frac{1}{2}[\arccos\sqrt{x} - \sqrt{x-x^2}\,],$$

$$\gamma(x) = \frac{1}{2}[\sqrt{x^2-x} - \operatorname{arccosh}\sqrt{x}\,].$$

Then

$$(-1)^n 2^\alpha e^{-2\kappa x} L_n^\alpha(4\kappa x) \sim Ai\left(\nu^{2/3}B^2\right) \sum_{k=0}^\infty \alpha_{2k}\nu^{-2k-1/3} - Ai'\left(\nu^{2/3}B^2\right) \sum_{k=0}^\infty \beta_{2k+1}\nu^{-2k-5/3},$$

as $n \to \infty$, uniformly for $x \in (b, \infty]$, where $b \in (0,1)$. $Ai(z)$ is the Airy function. The coefficients $\alpha_{2k}, \beta_{2k+1}$ follow from the recursion

$$h_k(u) = \alpha_k + \beta_k u + (u^2 - B^2) g_k(u), \quad h_{k+1}(u) = g_k'(u), \tag{7.4}$$

with

$$h_0(u) = h(u) = [1 - z^2(u)]^{(\alpha-1)/2} \frac{dz}{du},$$

where the relation between $z$ and $u$ is given by

$$\frac{1}{2}[\operatorname{arctanh} z - xz] = \frac{1}{3}u^3 - B^2(x)u.$$

The coefficients $\alpha_k, \beta_k$ are computed by substitution of $u = \pm B$ in (7.4); $B$ is defined in (7.3).

When $\alpha$ is also allowed to grow other expansions are needed (see [31] for expansions in the zeros domain in terms of elementary functions and [91] for expansions in terms of Bessel functions and Hermite polynomials).

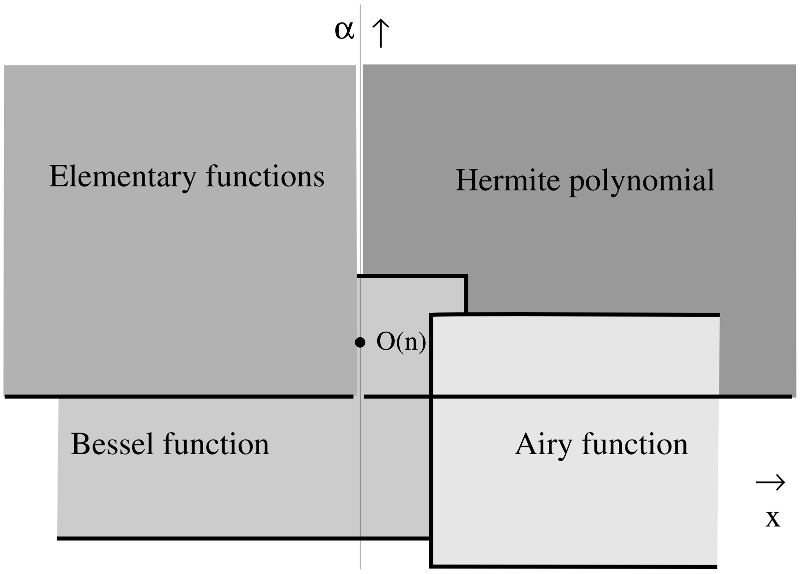


**Figure 7.2.** When $\alpha$ and $n$ of $L_n^\alpha(x)$ are both large and positive, 4 types of uniform asymptotic expansions and 4 domains are needed to cover the $(x, \alpha)$–half plane

*7.2. Fermi-Dirac integral*

Let

$$F_q(x) := \frac{1}{\Gamma(q+1)} \int_0^\infty \frac{t^q}{1 + e^{t-x}}\, dt, \quad q > -1.$$

These functions occur in quantum mechanics, for instance in problems on the distribution of particles that follow the Fermi–Dirac statistics. The problem is to find an asymptotic expansion for large values of $x$ which is uniformly valid with respect to $q \in [0, \infty)$. For details we refer to [68] and [95].

When $x < 0$ we have the convergent expansion:

$$F_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{xn}}{n^{q+1}}$$

When $x > 0$ we can transform into the contour integral

$$F_q(x) = \frac{1}{2i} \int_{\mathcal{L}} \frac{e^{xs}}{s^{q+1} \sin \pi s} \, ds,$$

where, initially, $\mathcal{L}$ is a vertical that cuts the real $s-$axis between 0 and 1.

The saddle point $s_0$ of the dominant part is the derivative of $\exp(xs - q \ln s)$, that is, $s_0 = q/x$. Translate the contour to pass through the point $s_0$ and sum the residues. Assuming $s_0 \neq 1, 2, 3, , \ldots$, we obtain:

$$F_q(x) = \sum_{n=1}^{N-1} \frac{(-1)^{n-1} e^{xn}}{n^{q+1}} + \frac{1}{2i} \int_{\mathcal{L}} \frac{e^{xs}}{s^{q+1} \sin \pi s} \, ds,$$

where $N$ is the integer satisfying $N - 1 < s_0 < N$ and $\mathcal{L}$ cuts the real positive axis at $s_0$, the saddle point.

To separate the pole near the saddle point write

$$h_N(s) = \frac{\pi}{\sin \pi s} - \frac{(-1)^N}{s - N}, \quad N = 0, 1, 2, \ldots .$$

Then

$$F_q(x) = \sum_{n=1}^{N-1} \frac{(-1)^{n-1} e^{xn}}{n^{q+1}} + G_q(x) + H_q(x),$$

$$G_q(x) = \frac{(-1)^N}{2\pi i} \int_{\mathcal{L}} \frac{e^{xs}}{s^{q+1}(s - N)} \, ds = \frac{(-1)^N e^{xN}}{N^{q+1}} Q(q + 1, xN),$$

$$H_q(x) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{e^{xs}}{s^{q+1}} h_N(s) \, ds$$

where $Q$ is the incomplete gamma function ratio:

$$Q(a, z) = \frac{1}{\Gamma(a)} \int_z^{\infty} t^{a-1} e^{-t} \, dt.$$

By expanding $h_N(s)$ at the saddle point:

$$h_N(s) = \sum_{n=0}^{\infty} c_n(s_0)(s - s_0)^n$$

we obtain:

$$H_q(x) \sim \frac{x^q}{\Gamma(q + 1)} \sum_{n=0}^{\infty} c_n(s_0) \phi_n(q) x^{-n}$$

where $\phi_n$ are simple polynomials defined by

$$\phi_n(q) = \frac{\Gamma(q+1)}{2\pi i} \int_{\mathcal{L}} \frac{e^t (t-q)^n}{t^{q+1}}\, dt$$

$$\phi_0(q) = 1, \quad \phi_1(q) = 0.$$

Higher $\phi_n$'s can be obtained by recursion.

*7.3. Stirling numbers*

The Stirling numbers of the first and second kind, denoted by $S_n^{(m)}$ and $\mathfrak{S}_n^{(m)}$, respectively, are defined as the coefficients in the expansions

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m,$$

$$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1).$$

The problem is to find the asymptotic behaviour of the Stirling numbers as $n \to \infty$, uniformly with respect to $m \in [0, n]$.

Alternative generating functions:

$$\frac{[\ln(x+1)]^m}{m!} = \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}, \quad \frac{(e^x-1)^m}{m!} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}.$$

This gives the Cauchy type integrals

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!} \frac{1}{2\pi i} \oint \frac{(e^x-1)^m}{x^{n+1}}\, dx$$

$$(-1)^{n-m} S_{n+1}^{(m+1)} = \frac{1}{2\pi i} \oint \frac{(x+1)(x+2)\cdots(x+n)}{x^{m+1}}\, dx.$$

Saddle point methods, suitably modified, give the uniform expansions; see [94].

*7.4. Incomplete gamma functions*

The definitions are

$$\gamma(a,z) = \int_0^z t^{a-1} e^{-t}\, dt, \quad \Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t}\, dt$$

with normalizations

$$P(a,z) = \frac{\gamma(a,z)}{\Gamma(a)}, \quad Q(a,z) = \frac{\Gamma(a,z)}{\Gamma(a)}.$$

Then $P(a,z) + Q(a,z) = 1$. Two methods discussed earlier:

- statistical distribution functions (see (6.3))
- pole near saddle point (see (6.1))

give the representations

$$\begin{aligned}
Q(a,z) &= \tfrac{1}{2}\,\mathrm{erfc}(\eta\sqrt{a/2}\,) + R_a(\eta),\\
P(a,z) &= \tfrac{1}{2}\,\mathrm{erfc}(-\eta\sqrt{a/2}\,) - R_a(\eta),\\
\eta &= (\lambda-1)\sqrt{2\frac{\lambda-1-\ln\lambda}{(\lambda-1)^2}}, \quad \lambda = \frac{z}{a},\\
R_a(\eta) &\sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\sum_{n=0}^{\infty}\frac{c_n(\eta)}{a^n}, \quad a\to\infty
\end{aligned}$$

uniformly with respect to $\eta \in \mathbb{R}$, or $z \geq 0$. For details we refer to [83]. This reference also admits complex values of $a$ and $z$.

## 8. The Stokes phenomenon

The Stokes phenomenon concerns the abrupt change across certain rays in the complex plane, known as *Stokes lines*, exhibited by the coefficients multiplying exponentially subdominant terms in compound asymptotic expansions. There is much recent interest in the Stokes phenomenon, and it fits in the present paper because it has to do with sudden changes in approximations when a certain parameter (in this case the phase of the large parameter) passes critical values. A complete discussions will not be given here. For a recent survey paper on the Stokes phenomenon we refer to [59].

### *8.1. The Airy function*

First we explain this phenomenon by using a simple example from differential equations. Consider Airy's equation

$$\frac{d^2y}{dz^2} = z\,y(z), \tag{8.1}$$

the solutions of which are entire functions. When $|z|$ is large the solutions of (8.1) are approximated by linear combinations of

$$u_\pm = z^{-\frac{1}{4}}\,e^{\pm\xi}, \quad \xi = \tfrac{2}{3}z^{3/2}. \tag{8.2}$$

Obviously, $u_\pm$ are multivalued functions of the complex variable $z$ with a branch point at $z=0$. Therefore, as we go once around the origin, the solutions of (8.1) will return to their original values, but $u_\pm$ will not. It follows that the constants $c_\pm$ in the linear combination

$$y(z) \sim c_- u_-(z) + c_+ u_+(z), \quad z\to\infty,$$

are domain-dependent. The constants change when we cross certain lines, the boundaries of certain sectors in the $z-$plane.

In the above example one of the terms $e^{\xi}, e^{-\xi}$ maximally dominates the other one at the rays $\arg z = 0, \arg z = \pm 2\pi/3$. In this example these 3 rays are the Stokes lines. At the rays $\arg z = \pm\pi/3$ and the negative $z-$axis the quantity $\xi$ is purely imaginary, and, hence, the terms $e^{\xi}, e^{-\xi}$ are equal in maginitude. These three rays are called the *anti-Stokes lines* [3]).

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[3]) This terminology is not the same in all branches of applied mathematics and mathematical physics: sometimes one sees a complete interchange of the names 'Stokes line' and 'anti-Stokes line'.

For the Airy function $Ai(z)$ we have the full asymptotic expansion (see [50], page 116)

$$Ai(z) \sim c_- z^{-\frac{1}{4}} e^{-\xi} \sum_{n=0}^{\infty} (-1)^n c_n \xi^{-n} \quad c_- = \frac{1}{2}\pi^{-\frac{1}{2}}, \quad |\arg z| < \frac{2}{3}\pi, \tag{8.3}$$

with coefficients

$$c_n = \frac{\Gamma(3n+\frac{1}{2})}{54^n\, n!\Gamma(n+\frac{1}{2})}, \quad n = 0, 1, 2, \ldots.$$

On the other hand, in an other sector of the $z$−plane, we have

$$Ai(z) \sim c_-\, z^{-\frac{1}{4}} \left[ e^{-\xi} \sum_{n=0}^{\infty} (-1)^n c_n \xi^{-n} + i e^{\xi} \sum_{n=0}^{\infty} c_n \xi^{-n} \right], \tag{8.4}$$

in which exactly the same term (with the same constant $c_-$) is involved as in (8.3), and there is another term corresponding with $u_+$. We can rewrite this in a more familiar expansion

$$Ai(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \left[ \sin(\xi + \tfrac{1}{4}\pi) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\xi^{2n}} - \cos(\xi + \tfrac{1}{4}\pi) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\xi^{2n+1}} \right], \tag{8.5}$$

in the sector $|\arg z| < \frac{2}{3}\pi$. In the overlapping domain of expansions (8.3) and (8.5), that is, when $\frac{1}{3}\pi < |\arg z| < \frac{2}{3}\pi$, the term with $u_+$ is asymptotically small compared with $u_-$, and it suddenly appears in the asymptotic approximation when we cross with increasing values of $|\arg z|$ the Stokes lines at $\arg z = \pm\frac{2}{3}\pi$. It seems that, when going from (8.3) to (8.4), the constant multiplying $u_+$ changes discontinuously from zero values (when $|\arg z| < \frac{2}{3}\pi$) to a non-zero value when we cross the Stokes line. This sudden appearance of the term $u_+$ does not have much influence on the asymptotic behaviour near the Stokes lines at $|\arg z| = \frac{2}{3}\pi$, because $u_+$ is dominated maximally by $u_-$ at these rays. However, see §8.3 below.

*8.2. The recent interest in the Stokes phenomenon*

This phenomenon of the *discontinuity of the constants* was discovered by Stokes and was discussed by him in a series of papers (on Airy functions in 1857, on Bessel functions in 1868). It is a phenomenon which is not confined to Airy or Bessel functions. The discovery by Stokes was, as Watson says, apparently one of those which are made at three o'clock in the morning. Stokes wrote in a 1902 retrospective paper: "The inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficients changed."

In 1989 the mathematical physicist Michael Berry provided a deeper explanation. He suggested that the coefficients of the subdominant expansion should be regarded not as a discontinuous constant but, for fixed $|z|$, as a continuous function of $\arg z$. Berry's innovative and insightful approach was followed by a series of papers by himself and other writers. In particular, Olver put the formal approach by Berry on a rigorous footing in papers with applications to confluent hypergeometric functions (including Airy functions, Bessel functions, and Weber parabolic functions).

At the same time interest arose in earlier work by Stieltjes, Airey, Dingle,... to expand remainders of asymptotic expansions at optimal values of the summation variable. This resulted in *exponentially-improved* asymptotic expansions, a method of improving asymptotic approximations as we have met in §1.3 in our introduction to Stieltjes work on asymptotics.

*8.3. Exponentially small terms in the Airy expansions*

We conclude this discussion by pointing out the relation between the Stokes phenomenon and the exponentially small terms in the asymptotic expansion of the Airy function. Consider the terms in the expansions in (8.3)–(8.5). They have the asymptotic form

$$c_n\xi^{-n} = \mathcal{O}\left[\Gamma(n)\,(2\xi)^{-n}\right], \quad n\to\infty.$$

When $z$ is large the terms decrease at first and then increase. The least term of the first series of (8.4) is near $n = n^* = [|2\xi|]$ and its size is of order $e^{-2|\xi|}$. At the Stokes lines at $|\arg z| = 2\pi/3$ the quantity $\xi$ is negative and the exponential term in front of the first series in (8.4) equals $e^{|\xi|}$. Hence the order of magnitude of $e^{-\xi}c_{n^*}\xi^{-n^*}$ is roughly of the same size as the second part in (8.5), that is, of the size of $e^{\xi}$ that is present in front of the second series. It follows that near the Stokes lines (and of course when $z$ turns to the negative axis) the second series in (8.5) is not at all negligible when we truncate the first series at the least term with index $n^*$.

At present we know, after Berry's observations, that near the Stokes lines one of the constants $c_\pm$ in the asymptotic representation in (8.2) in fact is a rapidly changing function of $z$. In the case of (8.4) we can write

$$Ai(z) \sim c_-\, z^{-\frac{1}{4}}\left[e^{-\xi}\sum_{n=0}^{\infty}(-1)^n c_n\xi^{-n} + iS(z)\,e^{\xi}\sum_{n=0}^{\infty}c_n\xi^{-n}\right],$$

where $S(z)$ switches rapidly but smoothly from 0 to 1 across the Stokes line at $\arg z = 2\pi/3$. As Berry observed (see [3]), a good approximation to $S(z)$ involves the error function, which function can describe the fast transition in this asymptotic problem. We have met the error function also in the cases (6.1) and (6.3) of Table 6.1, where it is used to describe similar fast transitions.

Many other writers have contributed recently in this field. In the Bibliography we have included recent papers concentrating on integrals, but the research is also making much progress in the area of differential equations. In [43] an introduction to the Stokes phenomenon is given from the view point of differential equations; for more recent developments (also in connection with exponentially-improved asymptotic expansions) we refer to [8], [48] and [54]. For integrals we refer to [3]–[14], [19], [34], [37], [46], [47], [51]–[53], [57]–[59].

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